Functions

A function takes an input, processes it and outputs a value. There are two main ways of writing a function: $f(x) = 5x + 2$ or $f: x \to 5x + 2$. Both of these say 'the function f takes a value for x, multiplies it by 5 and adds 2'. Functions can look a bit scary-mathsy, but they're just like equations but with y replaced by f(x).

Evaluating Functions (Grade 6)

This is easy — just shove the numbers into the function and you're away.

EXAMPLE: $f(x) = x^2 - x + 7$. Find a) f(3) and b) f(–2)

a) $f(3) = (3)^2 - (3) + 7 = 9 - 3 + 7 = 13$ b) $f(-2) = (-2)^2 - (-2) + 7 = 4 + 2 + 7 = 13$

Combining Functions (Grade 8)

1) You might get a question with two functions, e.g. f(x) and g(x), combined into a single function (called a composite function).
2) Composite functions are written e.g. fg(x), which means 'do g first, then do f' — you always do the function closest to x first.
3) To find a composite function, rewrite fg(x) as f(g(x)), then replace g(x) with the expression it represents and then put this into f.

Watch out — usually $fg(x) \neq gf(x)$. Never assume that they're the same.

EXAMPLE: If $f(x) = 2x - 10$ and $g(x) = -\frac{x}{2}$, find: a) fg(x) and b) gf(x).

a) $fg(x) = f(g(x)) = f(-\frac{x}{2}) = 2(-\frac{x}{2}) - 10 = -x - 10$

b) $gf(x) = g(f(x)) = g(2x - 10) = -\left(\frac{2x - 10}{2}\right) = -(x - 5) = 5 - x$

Inverse Functions (Grade 8)

The inverse of a function f(x) is another function, $f^{-1}(x)$, which reverses f(x). Here's the method to find it:

1) Write out the equation $x = f(y)$ ← f(y) is just the expression f(x), but with y's instead of x's
2) Rearrange the equation to make y the subject.
3) Finally, replace y with $f^{-1}(x)$.

EXAMPLE: If $f(x) = \frac{12 + x}{3}$, find $f^{-1}(x)$.

1) Write out $x = f(y)$: $x = \frac{12 + y}{3}$ ← So here you just rewrite the function replacing f(x) with x and x with y.
2) Rearrange to make y the subject: $3x = 12 + y$
$y = 3x - 12$
3) Replace y with $f^{-1}(x)$: $f^{-1}(x) = 3x - 12$

You can check your answer by seeing if $f^{-1}(x)$ does reverse f(x): e.g. $f(9) = \frac{21}{3} = 7$, $f^{-1}(7) = 21 - 12 = 9$

That page has really put the 'fun' into 'functions'...

Sorry, that joke just had to be made. This is another topic where practice really does make perfect.

Q1 If $f(x) = 5x - 1$, $g(x) = 8 - 2x$ and $h(x) = x^2 + 3$, find:

a) f(4)	[1 mark]	b) h(–2)	[1 mark]	c) gf(x)	[2 marks]
d) fh(x)	[2 marks]	e) gh(–3)	[2 marks]	f) $f^{-1}(x)$	[3 marks]

(Grade 8)

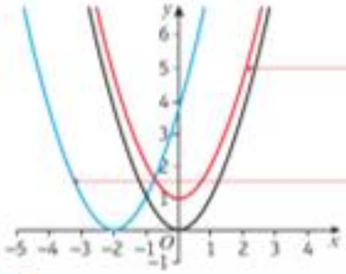
4.5 Translating graphs

You can transform the graph of a function by altering the function. Adding or subtracting a constant 'outside' the function translates a graph vertically.

- **The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\binom{0}{a}$.**

Adding or subtracting a constant 'inside' the function translates the graph horizontally.

- **The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\binom{-a}{0}$.**



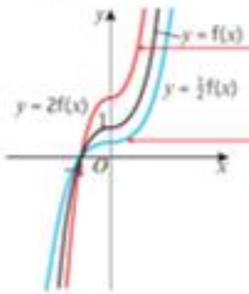
$y = f(x) + 1$ is a translation $\binom{0}{1}$, or 1 unit in the direction of the positive y-axis.

$y = f(x + 2)$ is a translation $\binom{-2}{0}$, or 2 units in the direction of the negative x-axis.

4.6 Stretching graphs

Multiplying by a constant 'outside' the function stretches the graph vertically.

- **The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a scale factor of a in the vertical direction.**

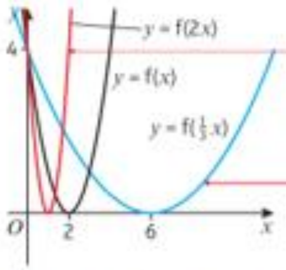


2f(x) is a stretch with scale factor 2 in the y-direction. All y-coordinates are doubled.

$\frac{1}{2}f(x)$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction. All y-coordinates are halved.

Multiplying by a constant 'inside' the function stretches the graph horizontally.

- **The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.**



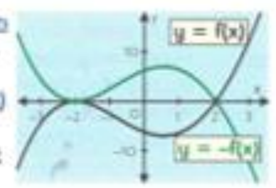
$y = f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction. All x-coordinates are halved.

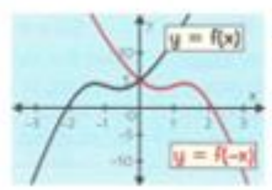
$y = f(\frac{1}{3}x)$ is a stretch with scale factor 3 in the x-direction. All x-coordinates are tripled.

Reflections: $y = -f(x)$ and $y = f(-x)$ (Grade 8)

$y = -f(x)$ is the reflection in the x-axis of $y = f(x)$.

Points (–2, 0) and (2, 0) are invariant points — they don't change during the transformation.





$y = f(-x)$ is the reflection in the y-axis of $y = f(x)$.

(0, 5) is the only invariant point under this reflection.

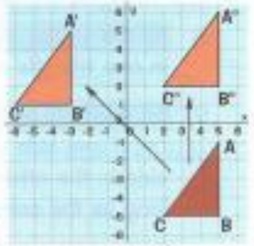
The Four Transformations

There are four transformations you need to know — translation, rotation, reflection and enlargement.

1) Translations (3)

In a translation, the amount the shape moves by is given as a vector (see p.103-104) written $\begin{pmatrix} x \\ y \end{pmatrix}$ — where x is the horizontal movement (i.e. to the right) and y is the vertical movement (i.e. up).
If the shape moves left and down, x and y will be negative. Shapes are congruent under translation (see p.78).

EXAMPLE:
a) Describe the transformation that maps triangle ABC onto A'B'C'.
b) Describe the transformation that maps triangle ABC onto A''B''C''.



a) To get from A to A', you need to move 8 units left and 6 units up, so...
The transformation from ABC to A'B'C' is a translation by the vector $\begin{pmatrix} -8 \\ 6 \end{pmatrix}$.

b) The transformation from ABC to A''B''C'' is a translation by the vector $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$.

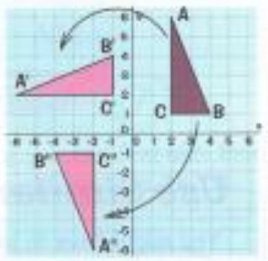
2) Rotations (3)

To describe a rotation, you must give 3 details:
1) The angle of rotation (usually 90° or 180°).
2) The direction of rotation (clockwise or anticlockwise).
3) The centre of rotation (often, but not always, the origin).

Shapes are congruent under rotation.

For a rotation of 180°, it doesn't matter whether you go clockwise or anticlockwise.

EXAMPLE:
a) Describe the transformation that maps triangle ABC onto A'B'C'.
b) Describe the transformation that maps triangle ABC onto A''B''C''.



a) The transformation from ABC to A'B'C' is a rotation of 90° anticlockwise about the origin.

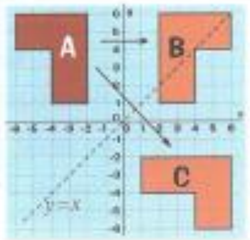
b) The transformation from ABC to A''B''C'' is a rotation of 180° clockwise (or anticlockwise) about the origin.

If it helps, you can use tracing paper to help you find the centre of rotation.

3) Reflections (3)

For a reflection, you must give the equation of the mirror line. Shapes are congruent under reflection as well.

EXAMPLE:
a) Describe the transformation that maps shape A onto shape B.
b) Describe the transformation that maps shape A onto shape C.



a) The transformation from A to B is a reflection in the y-axis.
b) The transformation from A to C is a reflection in the line $y = x$.

Points are invariant if they remain the same after a transformation — for reflections any point on the mirror line will be invariant.

Moving eet to ze left — a perfect translation...

The reason that shapes are congruent under translation, reflection and rotation is because their size and shape don't change, just their position and orientation. Now have a go at this question:

Q1 On a grid, copy shape A above and rotate it 90° clockwise about the point (–1, –1). [2 marks] (3)

The Four Transformations

One more transformation coming up — enlargements. They're the trickiest, but also the most fun (honest).

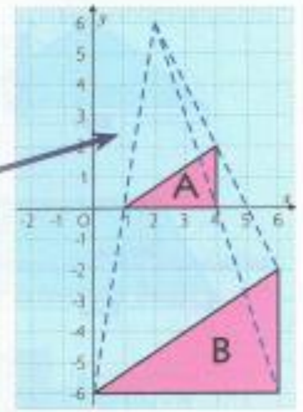
4) Enlargements (4)

For an enlargement, you must specify:
1) The scale factor.
2) The centre of enlargement.

$$\text{scale factor} = \frac{\text{new length}}{\text{old length}}$$

Shapes are similar under enlargement — the position and the size change, but the angles and ratios of the sides don't (see p.79).

EXAMPLE:
a) Describe the transformation that maps triangle A onto triangle B.
b) Describe the transformation that maps triangle B onto triangle A.



a) Use the formula above to find the scale factor (just choose one side): scale factor $= \frac{6}{3} = 2$

For the centre of enlargement, draw lines that go through corresponding vertices of both shapes and see where they cross.

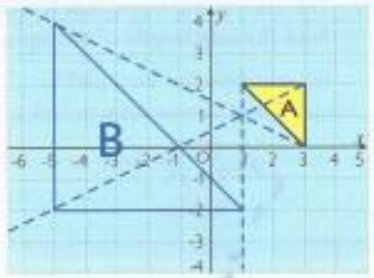
So the transformation from A to B is an enlargement of scale factor 2, centre (2, 6)

b) Using a similar method, scale factor $= \frac{3}{6} = \frac{1}{2}$ and the centre of enlargement is the same as before, so the transformation from B to A is an enlargement of scale factor $\frac{1}{2}$, centre (2, 6)

Scale Factors — Four Key Facts (6)

1) If the scale factor is bigger than 1 the shape gets bigger.
2) If the scale factor is smaller than 1 (e.g. ½) it gets smaller.
3) If the scale factor is negative then the shape pops out the other side of the enlargement centre. If the scale factor is –1, it's exactly the same as a rotation of 180°.
4) The scale factor also tells you the relative distance of old points and new points from the centre of enlargement — this is very useful for drawing an enlargement, because you can use it to trace out the positions of the new points.

EXAMPLE: Enlarge shape A below by a scale factor of -3, centre (1, 1). Label the transformed shape B.



1) First, draw lines going through (1, 1) from each vertex of shape A.
2) Then, multiply the distance from each vertex to the centre of enlargement by 3, and measure this distance coming out the other side of the centre of enlargement.
So on shape A, vertex (3, 2) is 2 right and 1 up from (1, 1) — so the corresponding point on shape B will be 6 left and 3 down from (1, 1). Do this for every point.
3) Join the points you've drawn to form shape B.

Scale factors — they're enough to put the fear of cod into you...

If you have to do more than one transformation, just do them one at a time — here's some practice.

Q1 On a grid, draw triangle A with vertices (2, 1), (4, 1) and (4, 3). Enlarge it by a scale factor of –2 about point (1, 1), then reflect it in the line $x = 0$. [3 marks] (6)